Aliquot sequences with explosive growth

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The goal of this document is to prove that there are aliquot sequences which, for a given pair ℓ, c of integers, increase by a factor at least c during at least ℓ consecutive iterations.

We denote by p_k the k^{th} prime. Let $t_1 = 1$ and, for all $k \geq 1$, put

$$t_{k+1} = \varphi(p_k^{1+t_k} \cdot (-1 + p_{k+1})) - 1$$

where φ designs Euler's indicator. For all $k \geq 1$, put $A_k = \{m \mid \forall 1 \leq i \leq k \ \mathrm{val}_{p_i}(m) = t_i\}$, i.e. the set of natural numbers whose factorization starts by $p_1^{t_1} \ldots p_k^{t_k}$. Finally, we note for all m, $s(m)$ the sum of divisors of m and $s'(m) := s(m) - m$.

The main result is a corollary of the following result of H.W.Lenstra.

Theorem 1. *([1]) For all $k \geq 2$ and all $m \in A_k$, $s'(m) \in A_{k-1}$.*

Proof. : Let $m \in A_k$ for some $k \geq 2$. Then $m = p_1^{t_1} \cdots p_k^{t_k} B$ for some integer B, relatively prime to $p_1 \cdots p_k$. Since the sum of the divisors is a multiplicative function, we have

$$s'(m) = s(p_1^{t_1}) \cdots s(p_k^{t_k}) s(B) - p_1^{t_1} \cdots p_k^{t_k} B. \tag{1}$$

Now recall that Fermat's little theorem says that for all n and a such that $\gcd(n, a) = 1$, $a^{\varphi(n)} \equiv 1 \mod n$. For each $i \in [1, k-1]$, we apply it for $a = p_{i+1}$ and $n = p_i^{1+t_i}(-1 + p_{i+1})$ to obtain

$$p_{i+1}^{\varphi((p_i^{1+t_i}(p_{i+1}-1))} - 1 \equiv 0 \mod p_i^{1+t_i}(p_{i+1} - 1), \tag{2}$$

which is equivalent to $p_{i+1}^{1+t_{i+1}} - 1 \equiv 0 \mod p_i^{1+t_i}(p_{i+1} - 1)$. Further it gives $\frac{p_{i+1}^{1+t_{i+1}} - 1}{p_{i+1} - 1} \equiv 0 \mod p_i^{1+t_i}$. Since $s(p_{i+1}^{t_{i+1}}) = \frac{p_{i+1}^{t_{i+1}+1} - 1}{p_{i+1} - 1}$, we obtain

$$p_i^{1+t_i} \mid s(p_{i+1}^{t_{i+1}}). \tag{3}$$

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By combining Equations (1) and (3) we get $p_i^{t_i} \mid s'(m)$ and $p_i^{1+t_i} \not| s'(m)$. So $s'(m) \in A_{k-1}$. $\square$

Example 1. *One computes* $t_1 = 1$, $t_2 = 4$ *and* $t_3 = 324$. *For all* m *divisible by* $2 \cdot 81 \cdot 5^{324}$, *one has* $2 \cdot 81 \mid s'(m)$ *and* $2 \mid s'(s'(m))$.

And now the new result we proved.

Corollary 1. *For all* $c > 0$ *and* $\ell \in \mathbb{N}$, *there is an aliquot sequence such that for* ℓ *consecutive iterations one has* $\frac{s'(m)}{m} > c$.

Proof. : Mertens' formula wites $\sum_{p \text{ prime}, p \leq x} \frac{1}{p} \sim \log\log x$. In particular, $\sum_{p \text{ prime}} \frac{1}{p}$ diverges and the product $\Pi_{p \text{ prime}}(1 + \frac{1}{p})$ goes to infinity. Hence there exists an $n \in \mathbb{N}$ such that $\Pi_{i=1}^{n}(1 + \frac{1}{p_i}) > c + 1$. Let m_0 be any element of $A_{n+\ell}$. By Lenstra's Theorem, the first $\ell + 1$ iterations of the aliquote sequence starting at m_0 belong to $A_{n+\ell}, A_{n+\ell-1}, \ldots, A_n$ respectively. Let $k \in [0, \ell]$. Then $m := (s')^k(m_0)$ belongs to $A_{n+\ell-k}$, so it factors as $p_1^{t_1} \cdots p_{n+\ell-k}^{t_{n+\ell-k}} B$ for some B relatively prime to $p_1, \ldots, p_{n+\ell-k}$. We obtain $\frac{s(m)}{m} = \frac{s(B)}{B} \Pi_{i=1}^{n+\ell-k}(1 + \frac{1}{p_i} + \cdots + \frac{1}{p_i^{t_i}}) \geq \frac{s(B)}{B} \Pi_{i=1}^{n+\ell-k}(1 + \frac{1}{p_i}) \geq \frac{s(B)}{B} \Pi_{i=1}^{n}(1 + \frac{1}{p_i})$, which is greater than $\frac{s(B)}{B}(c + 1)$ by the choice we made on n. Finally, $s(B)$ is the sum of divisors of B, including B, so $s(B) > B$. We obtain $\frac{s(m)}{m} > c + 1$ and, since $\frac{s'(m)}{m} = \frac{s(m)}{m} - 1$, $\frac{s'(m)}{m} > c$. $\square$

Example 2. *For* $c = 1$ *and* $\ell = 2$ *the corollary states that all the aliquot sequences starting at an element of* A_3, *i.e. divisible by* $2 \cdot 81 \cdot 5^{324}$, *will increase at least at the first two iterations.*

References

[1] On asymptotic properties of aliquot sequences, P. Erdos , Mathematics of Computation, volume 30, pages 641-645, 1976