Amicable pairs of Gaussian integers

Search methods and classification of 6,038 pairs

Summary note

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Abstract

We describe how 6,038 amicable pairs of Gaussian integers (in Spira's sense) were obtained: 967 "non-trivial" pairs coming from an exhaustive search, and 5,071 "trivial" pairs derived from them by exchanging factors, by te Riele's and Borho's breeders, and by a Gaussian analogue of the Thabit–Euler rule. All of them have been verified. We then propose a classification: a unit law with no exception, which splits the pairs into two families according to whether $\sigma(a)/a$ is close to $1-i$ or to $-2i$ (a being the common part); the type (i,j) and regularity; the families sharing a common part; and a striking geometric consequence: the two members of a pair are almost perpendicular or almost aligned.

1 Definitions

The units of $\mathbb{Z}[i]$ are $1, i, -1, -i$. A Gaussian prime π is called *normalized* if $\operatorname{Re}\pi > 0$ and $\operatorname{Im}\pi \geq 0$; every $z \neq 0$ can be written uniquely as $z = \varepsilon(z)\prod \pi^{e_\pi}$ with a unit $\varepsilon(z)$ and normalized primes. Following Spira (1961), we set

$$\sigma(z) = \prod_{\pi^e \,\|\, z} \left(1 + \pi + \pi^2 + \cdots + \pi^e\right), \qquad h(z) = \frac{\sigma(z)}{z} \in \mathbb{Q}(i),$$

and (z, w), $z \neq w$, is an *amicable pair* if $\sigma(z) = \sigma(w) = z + w$. The function σ does not depend on the unit $\varepsilon(z)$; it is not compatible with complex conjugation.

For a pair (z, w), the *common part* is $a = \prod \pi^{\min(e_\pi(z),\, e_\pi(w))}$ (a product of normalized primes), and we write

$$z = \varepsilon\, a\, M, \qquad w = \varepsilon'\, a\, N, \qquad \varepsilon = \varepsilon(z),\ \varepsilon' = \varepsilon(w),$$

where M and N are products of normalized primes. The pair is *regular* if every prime of a has the same exponent in z and in w (then a, M, N are pairwise coprime). Its *type* (i, j) is given by the numbers of distinct primes of M and of N, as in the classical tables of amicable pairs in $\mathbb{N}$.

2 The data and their verification

Each pair was verified independently of the programs that produced it: factorization of z and w into normalized primes; primality of the 13,080 distinct Gaussian primes involved (norms up to $7.2 \cdot 10^{27}$), certified by GNU `factor`, which proves primality with the Lucas test; product equal to z and to w up to a unit; $\sigma(z) = \sigma(w) = z + w$. No pair occurs twice (as an unordered pair) and no number belongs to two pairs. The check was tested on deliberate errors (flipped sign, repeated pair, fake prime factor).

Origin	Pairs	Category
Initial list, exhaustive search (10 sign errors corrected)	962	non-trivial
Added pairs, exhaustive search (1 sign error corrected)	5	non-trivial
Iteration from the 962 pairs (exchanges, breeders)	1,126	trivial
First run: 48 h on 120 threads	2,393	trivial
Second run: 48 h on 124 threads, targeted families	1,520	trivial
Gaussian Thabit–Euler rule (§ 3.4)	32	trivial
Total	6,038	967 + 5,071

Table 1: Origin of the pairs. A pair is called *trivial* if it can be deduced from already known pairs, including by a new method.

3 Generation methods

The programs (Python/SageMath and C) and the analyses were produced with the AI assistant Claude (Anthropic). Methods **E**, **F** and **B** are adaptations to $\mathbb{Z}[i]$ of the classical methods for $\mathbb{N}$; they are iterated: every new pair is fed back in.

3.1 Exchange of factors (method E, 1,365 pairs)

If t divides z and w with the same exponents, and if T, coprime to z/t and w/t, satisfies $h(T) = h(t)$, then $(zT/t,\ wT/t)$ is again amicable. For $t = \pi^k s$, one solution is $T = \pi^{k+j}\rho\, s$ with

$$\rho = -\frac{\sigma(\pi^{k+j})}{\sigma(\pi^{j-1})},$$

provided ρ is a normalized prime (impossible in $\mathbb{N}$, where ρ would be negative). The other solutions are sought in the form $T = Q\, s\, \rho$, where Q is the reduced denominator of $h(t)$ and ρ a prime given by a linear equation.

3.2 Breeders and te Riele's daughters (methods D and F, 28 pairs)

A *breeder* is a triple (a, u, x) such that $\sigma(u) = 1 + x$ and $\sigma(a)(1 + x) = a(\varepsilon_1 u + \varepsilon_2 x)$. It generates the pairs $(\varepsilon_1\, a\, u\, q,\ \varepsilon_2\, a\, r\, c)$ whenever

$$(r - x)(c - x) = (x + 1)(x + \eta u), \qquad q = r + c + \eta u, \qquad \eta = \varepsilon_1/\varepsilon_2,$$

with q, r, c normalized primes (and a similar formula for other pairs of units of the daughters). Method D detects the breeders hidden in known pairs ($z = \varepsilon_1\, a\, u\, q$).

3.3 Exhaustive search for Borho breeders (method B, 3,646 pairs)

With $u = m\pi$, the breeder equation can be solved for the unknown prime:

$$\pi = -\frac{\varepsilon_2\, a + \sigma(m)\, G}{\sigma(m)\, G - \varepsilon_1\, a\, m}, \qquad G = \sigma(a) - \varepsilon_2\, a.$$

A C program scans all pairs (a, m) in a given region (bounds on $N(a)$, on $N(m)$ and on the norm of the largest prime of a), for all 16 pairs of units. Since $|\sigma(\pi^e)/\pi^e| \geq 1$ for every normalized prime, the necessary condition

$$|h(a) - \varepsilon_2| \cdot |h(m)| \leq 1 + \frac{1}{\sqrt{2}\,|m|}$$

allows whole branches of the tree of m to be pruned. The first run scanned, region by region, the common parts all of whose primes have norm $\le 10^7$, up to $N(a) \le 10^{11}$ (with bounds on $N(m)$ that decrease as $N(a)$ grows). Its analysis showed that the common parts $a = (1+i)^7 pq$ or $(1+i)^8 pq$ (p, q primes of norm > 50) yielded 46% of the daughters for 0.1% of the computing time; the second run therefore targeted this family, the common parts with a single large prime, and a few fertile odd forms, up to $N(a) \le 10^{14}$ and $N(m) \le 10^6$.

3.4 A Gaussian Thabit–Euler rule (32 pairs)

The pairs of type $(2,1)$, $z = \varepsilon\, a\, p\, q$ and $w = \varepsilon'\, a\, r$, are the analogue of Thabit ibn Qurra's pairs such as $(220, 284)$. The condition $\sigma(p)\sigma(q) = \sigma(r)$ forces $r = pq + p + q$; setting $X = p+1$, $Y = q+1$ and $D = \sigma(a) - a(\varepsilon + \varepsilon')$, the condition $\sigma(z) = z + w$ becomes

$$(DX + a\varepsilon)(DY + a\varepsilon) = a\big[D(\varepsilon - \varepsilon') + a\varepsilon^2\big].$$

For a given common part a, it therefore suffices to run through the divisors of the right-hand side. For example $z = -i\,(1+i)^8 \cdot 23 \cdot (45+2i)$, $w = (1+i)^8(1103+48i)$, with $1103 + 48i = 23(45+2i) + 23 + (45+2i)$. Applied to the 544 known common parts and to $(1+i)^n$, $n \le 60$, this rule recovered the 5 already known pairs of type $(2,1)$ and produced 32 new pairs. Borho breeders cannot generate this type.

4 Classification

4.1 A unit law

In all 6,038 pairs, the unordered pair $\{\varepsilon(z), \varepsilon(w)\}$ is always $\{-i, 1\}$ or $\{-i, -i\}$:

	non-trivial	trivial	total
$\{\varepsilon, \varepsilon'\} = \{-i, 1\},\ \varepsilon + \varepsilon' = 1 - i$	521	3,178	3,699
$\{\varepsilon, \varepsilon'\} = \{-i, -i\},\ \varepsilon + \varepsilon' = -2i$	446	1,893	2,339
other pairs of units	0	0	0

Proposition. *For a regular pair $(z, w) = (\varepsilon a M,\ \varepsilon' a N)$, we have $\sigma(M) = \sigma(N)$ and*

$$h(a) = \frac{\varepsilon}{h(M)} + \frac{\varepsilon'}{h(N)}.$$

Indeed $\sigma(z) = \sigma(a)\sigma(M)$, $\sigma(w) = \sigma(a)\sigma(N)$ and $\sigma(a)\sigma(M) = a(\varepsilon M + \varepsilon' N)$. This is the analogue of $\sigma(a)/a = M/\sigma(M) + N/\sigma(N)$ in $\mathbb{N}$; the identity was checked in exact arithmetic on the 5,900 regular pairs.

Lemma. *For every normalized prime π and every $e \ge 1$, $\operatorname{Re} h(\pi^e) > 0$ and $\operatorname{Im} h(\pi^e) \le 0$, with equality only for the inert primes $q \equiv 3 \pmod 4$.*

For $N(\pi) \ge 13$, write $\pi = re^{i\theta}$ with $0^\circ \le \theta < 90^\circ$; since $|\sin k\theta| \le k \sin\theta$,

$$-\operatorname{Im} h(\pi^e) = \sum_{k=1}^{e} r^{-k} \sin k\theta \ \ge\ r^{-1}\sin\theta\left(2 - \frac{1}{(1-r^{-1})^2}\right) \ \ge\ 0,$$

and $\operatorname{Re} h(\pi^e) \ge 1 - \frac{1}{r-1} > 0$; the cases $1+i$, $2+i$, $1+2i$ and 3 are checked directly.

Each factor $h(\pi^e)$ therefore turns clockwise, and $h(a)$ moves from 1 into the lower half-plane. When M and N consist of large primes, $h(M) \approx h(N) \approx 1$ and the proposition gives $h(a) \approx \varepsilon + \varepsilon'$:

the targets that common parts of reasonable size can reach are $1-i$ and $-2i$. On average we observe $h(a) = 0.999 - 0.945\,i$ for the first family and $0.111 - 1.843\,i$ for the second; for the trivial pairs, the median distance $|h(a) - (\varepsilon + \varepsilon')|$ is 0.014 and 0.098 respectively. This explains the fertility of the powers of $1+i$: $h\big((1+i)^k\big) = 1 - i + i(1+i)^{-k}$ tends to $1-i$ (for instance $h((1+i)^8) = 1 - \frac{15}{16}i$). A pair with units $\{1, 1\}$ would require $h(a) \approx 2$, hence a common part made almost entirely of inert primes, such as $a = 3^5 \cdot 7^3 \cdot 11^2 \cdot 19$ ($h(a) \approx 2.02$, $N(a) \approx 3.7 \cdot 10^{16}$), outside the explored range. The unit law is therefore empirical; it has not been proved.

4.2 Type, regularity, exponent of $1+i$

type (i,j)	(3,2)	(4,2)	(3,3)	(4,3)	(4,4)	(5,4)	(2,2)	(2,1)	(5,3)	others
non-trivial	131	13	297	209	148	84	43	4	18	20
trivial	3,739	839	127	129	146	42	7	33	6	3

Table 2: Distribution by type (i,j) (numbers of distinct primes of M and of N, the larger one first).

- Breeders' daughters are almost all of type $(3,2)$ or $(4,2)$, since $N = rc$; the non-trivial pairs are more varied, mostly $(3,3)$, $(4,3)$ and $(4,4)$.
- 5,900 pairs are regular (858 non-trivial, 5,042 trivial); M and N are squarefree in 98% of the cases.
- The exponent of $1+i$ is the same in z and in w for 6,029 pairs. The 9 exceptions are non-trivial, for example $z = (1+i)^9 \cdots$, $w = -i\,(1+i)^7 \cdots$: they are the analogues of pairs in $\mathbb{N}$ such as $(1184, 1210)$. The most frequent common exponents are 8, 7, 0, 15 and 6.

4.3 Families sharing a common part, and abundancy classes

The regular pairs have 544 distinct common parts. The largest families are $a = (1+i)^{15}(308+377i)$ (228 pairs, daughters of a single breeder), $a = (1+2i)^2(2+5i)(1+10i)(10+9i)$ (166) and $a = (1+i)^8(13+22i)(147+8i)$ (132); $a = (1+i)^8$ alone is the common part of 97 pairs, 95 of which are non-trivial.

By the proposition, two regular pairs with the same "skeleton" $(\varepsilon, \varepsilon', M, N)$ have common parts with the same abundancy $h(a)$; conversely, any common part a' with $h(a') = h(a)$ that is coprime to MN gives an amicable pair $(\varepsilon a' M,\ \varepsilon' a' N)$. The pairs thus fall into 367 abundancy classes, each being the product of its common parts and its skeletons. The list is *closed* under this operation: no compatible combination is missing. On the other hand, no mirror symmetry appears: replacing each prime $a+bi$ by $b+ai$ never yields an amicable pair.

5 The angle between the two members of a pair

For a regular pair, $\sigma(M) = \sigma(N)$ reads $M\,h(M) = N\,h(N)$, hence the exact equality

$$\frac{w}{z} = \frac{\varepsilon'}{\varepsilon} \cdot \frac{h(M)}{h(N)}.$$

When M and N consist of large primes, $h(M) \approx h(N) \approx 1$, so $w/z \approx \varepsilon'/\varepsilon$. For the units $\{-i, 1\}$, $\varepsilon'/\varepsilon = \pm i$: the two members are almost *perpendicular*. For the units $\{-i, -i\}$, $\varepsilon'/\varepsilon = 1$: they are almost *aligned*. In both cases $|w| \approx |z|$.

All angles lie between 63° and 100.2° for the first family and between 0° and 28° for the second. For a regular pair, the deviation from 90° or from 0° is exactly the argument of $h(M)/h(N)$: it is larger for the non-trivial pairs, whose M and N contain small primes.

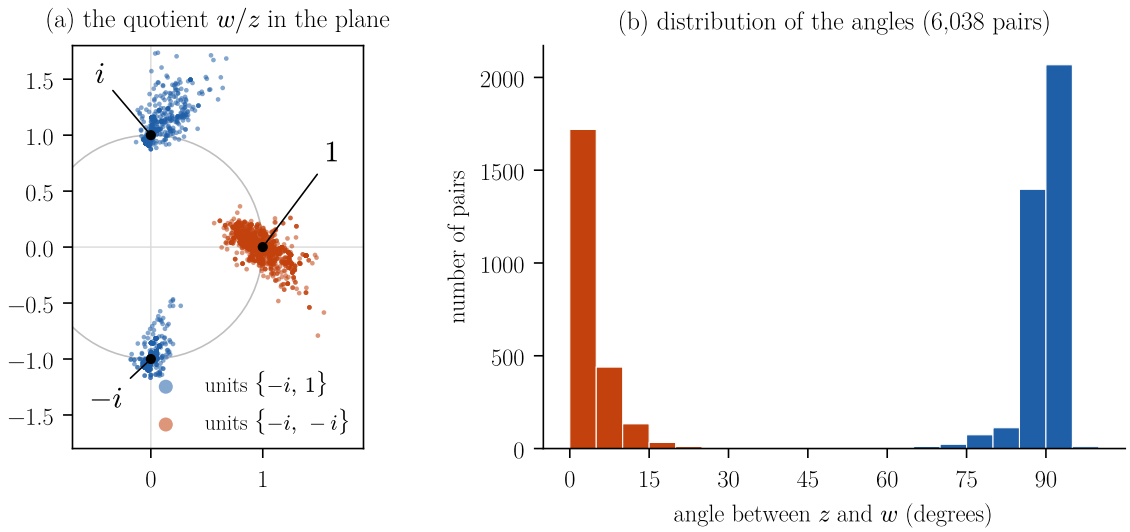

Figure 1: (a) The quotient w/z (z being the member with the smaller real part) clusters near i, $-i$ and 1. (b) Angle between z and w for the 6,038 pairs.

units	pairs	median angle	80% of the pairs between	median ratio of moduli
$\{-i, 1\}$	trivial	90.05°	88.9° and 91.4°	1.010
$\{-i, 1\}$	non-trivial	88.55°	76.5° and 92.4°	1.069
$\{-i, -i\}$	trivial	1.29°	0.1° and 6.8°	1.046
$\{-i, -i\}$	non-trivial	5.28°	0.8° and 12.9°	1.118

6 Open questions

- Prove the unit law, or find a pair with other units (for instance $\{1, 1\}$, which requires $h(a) \approx 2$).
- Explore pairs of type $(2, 1)$ systematically with the Thabit–Euler rule, for other common parts close to the targets $1 - i$ and $-2i$.
- Extend the search for Borho breeders: in the second run, the regions $10^5 < N(m) \leq 10^6$ produced 1,014 of the 1,520 new pairs.

References

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